

New Developments in Ostrowski's Type Inequalities by Using 13-Step Quadratic Kernel

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Abstract: The Ostrowski inequality has recently been widely recognized as a powerful mathematical tool for modeling and analysing many physical and real-world events. This inequality has been applied across diverse mathematical fields, including complex analysis, numerical analysis, functional analysis, and stochastic analysis. The purpose of this paper is to develop Ostrowski's type integral inequalities by using a special type of 13-steps quadratic kernel. The incorporation of this new kernel gives some new error bounds for various quadrature rules. Applications for cumulative distributive functions are also discussed.

Keywords: Ostrowski inequalities, numerical integration, quadratic kernel, cumulative distributive functions.

1. Introduction

Integral inequalities play a crucial role in various mathematical applications. These inequalities have gained significant prominence in recent years, evolving into a distinct field of study within mathematics. This evolution has led to significant advancements and deeper insights, enhancing our understanding and application of integral inequalities in complex mathematical and real-world problems. Notably, Ostrowski [1] discovered the integral inequality now known as the "Ostrowski inequality" in 1938. Inequalities and their applications have been extensively discussed in various research publications and articles (for example, [2-6]). The primary objective of our research is to develop generalized results using a novel approach "a 13-step quadratic kernel" that extends previously published findings in [7-12]. Recent work by various authors (e.g., [13-17]) has further utilized these developments to establish numerous valuable and noteworthy inequalities. Furthermore, we explore different results by utilizing the Diaz-Metcalf inequality, Cauchy inequality, and Grüss inequality. Finally, these inequalities are applied to the cumulative distribution function.

2. Main Findings

Before presenting our main results, we must first establish the following lemma.

Lemma: Let $Y: [r, s] \rightarrow \mathbb{R}$ be such that Y' is absolutely continuous on $[r, s]$. Define the kernel $P(t, \tilde{u})$ as

$$P(t, \tilde{u}) = \begin{cases} \frac{1}{2}(\tilde{u} - r)^2 & ; \quad \tilde{u} \in (r, \frac{31r+t}{32}] \\ \frac{1}{2}(\tilde{u} - \frac{63r+s}{64})^2 & ; \quad \tilde{u} \in (\frac{31r+t}{32}, \frac{15r+t}{16}] \\ \frac{1}{2}(\tilde{u} - \frac{31r+s}{32})^2 & ; \quad \tilde{u} \in (\frac{15r+t}{16}, \frac{7r+t}{8}] \\ \frac{1}{2}(\tilde{u} - \frac{15r+s}{16})^2 & ; \quad \tilde{u} \in (\frac{7r+t}{8}, \frac{3r+t}{4}] \\ \frac{1}{2}(\tilde{u} - \frac{7r+s}{8})^2 & ; \quad \tilde{u} \in (\frac{3r+t}{4}, \frac{r+t}{2}] \\ \frac{1}{2}(\tilde{u} - \frac{3r+s}{4})^2 & ; \quad \tilde{u} \in (\frac{r+t}{2}, t] \\ \frac{1}{2}(\tilde{u} - \frac{r+s}{2})^2 & ; \quad \tilde{u} \in (t, r+s-t] \\ \frac{1}{2}(\tilde{u} - \frac{r+3s}{4})^2 & ; \quad \tilde{u} \in (r+s-t, \frac{r+2s-t}{2}] \\ \frac{1}{2}(\tilde{u} - \frac{r+7s}{8})^2 & ; \quad \tilde{u} \in (\frac{r+2s-t}{2}, \frac{r+4s-t}{4}] \\ \frac{1}{2}(\tilde{u} - \frac{r+15s}{16})^2 & ; \quad \tilde{u} \in (\frac{r+4s-t}{4}, \frac{r+8s-t}{8}] \\ \frac{1}{2}(\tilde{u} - \frac{r+31s}{32})^2 & ; \quad \tilde{u} \in (\frac{r+8s-t}{8}, \frac{r+16s-t}{16}] \\ \frac{1}{2}(\tilde{u} - \frac{r+63s}{64})^2 & ; \quad \tilde{u} \in (\frac{r+16s-t}{16}, \frac{r+32s-t}{32}] \\ \frac{1}{2}(\tilde{u} - s)^2 & ; \quad \tilde{u} \in (\frac{r+32s-t}{32}, s] \end{cases} \quad (2.1)$$

$$\forall t \in [r, \frac{r+s}{2}].$$

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Proof:

Integrating by parts, we have the following identity

$$\begin{aligned}
 & \int_r^{\check{s}} P(t, \check{u}) Y''(\check{u}) d\check{u} \\
 &= \int_r^{\check{s}} Y(\check{u}) d\check{u} - \frac{\check{s} - r}{4} \left[Y(t) + Y(r + \check{s} - t) + \frac{1}{2} \left\{ Y\left(\frac{r+t}{2}\right) + Y\left(\frac{r+2\check{s}-t}{2}\right) \right\} \right. \\
 & \quad + \frac{1}{4} \left\{ Y\left(\frac{3r+t}{4}\right) + Y\left(\frac{r+4\check{s}-t}{4}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{7r+t}{8}\right) + Y\left(\frac{r+8\check{s}-t}{8}\right) \right\} \\
 & \quad + \frac{1}{16} \left\{ Y\left(\frac{15r+t}{16}\right) + Y\left(\frac{r+16\check{s}-t}{16}\right) + Y\left(\frac{31r+t}{32}\right) + Y\left(\frac{r+32\check{s}-t}{32}\right) \right\} \\
 & \quad + \left(t - \frac{5r+3\check{s}}{8} \right) \left\{ 2\{Y(r+\check{s}-t) - Y(t)\} + \frac{1}{2} \left\{ Y\left(\frac{r+2\check{s}-t}{2}\right) \right. \right. \\
 & \quad \left. \left. - Y\left(\frac{r+t}{2}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{r+4\check{s}-t}{4}\right) - Y\left(\frac{3r+t}{4}\right) \right\} + \frac{1}{32} \left\{ Y\left(\frac{r+8\check{s}-t}{8}\right) \right. \right. \\
 & \quad \left. \left. - Y\left(\frac{7r+t}{8}\right) \right\} + \frac{1}{128} \left\{ Y\left(\frac{r+16\check{s}-t}{16}\right) - Y\left(\frac{15r+t}{16}\right) \right\} \right\} \\
 & \quad + \frac{1}{512} \left(t - \frac{3r+\check{s}}{4} \right) \left\{ Y\left(\frac{r+32\check{s}-t}{32}\right) - Y\left(\frac{31r+t}{32}\right) \right\}. \tag{2.2}
 \end{aligned}$$

Now, we are going to use following three different conditions on Y'' and Y''' to present our results.

Case 1: For $Y'' \in L^1[r, \check{s}]$

Theorem 1. Let $Y: [r, \check{s}] \rightarrow \mathbb{R}$ be differentiable on (r, \check{s}) , Y' is absolutely continuous on

$[r, \check{s}]$ and $\gamma \leq Y''(\check{u}) \leq \Gamma, \forall \check{u} \in [r, \check{s}]$. Then $\forall t \in \left[r, \left(\frac{r+\check{s}}{2}\right) \right]$, we get

$$\begin{aligned}
 & \left| \frac{1}{4} \left[Y(t) + Y(r + \check{s} - t) + \frac{1}{2} \left\{ Y\left(\frac{r+t}{2}\right) + Y\left(\frac{r+2\check{s}-t}{2}\right) \right\} + \frac{1}{4} Y\left(\frac{3r+t}{4}\right) \right. \right. \\
 & \quad \left. \left. + Y\left(\frac{r+r-t}{4}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{7r+t}{8}\right) + Y\left(\frac{r+8\check{s}-t}{8}\right) \right\} + \frac{1}{16} \left\{ Y\left(\frac{15r+t}{16}\right) \right. \right. \\
 & \quad \left. \left. + Y\left(\frac{r+16\check{s}-t}{16}\right) + Y\left(\frac{31r+t}{32}\right) + Y\left(\frac{r+32\check{s}-t}{32}\right) \right\} + \left(t - \frac{5r+3\check{s}}{8} \right) \{ 2Y(r+\check{s}-t) \right. \right. \\
 & \quad \left. \left. - Y(t) \right\} + \frac{1}{2} \left\{ Y\left(\frac{r+2\check{s}-t}{2}\right) - Y\left(\frac{r+t}{2}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{r+4\check{s}-t}{4}\right) - Y\left(\frac{3r+t}{4}\right) \right\} \right. \\
 & \quad \left. + \frac{1}{32} \left\{ Y\left(\frac{r+8\check{s}-t}{8}\right) - Y\left(\frac{7r+t}{8}\right) \right\} + \frac{1}{128} \left\{ Y\left(\frac{r+16\check{s}-t}{16}\right) - Y\left(\frac{15r+t}{16}\right) \right\} \right\} \\
 & \quad + \frac{1}{512} \left(t - \frac{3r+\check{s}}{4} \right) \left\{ Y\left(\frac{r-32\check{s}-t}{32}\right) - Y\left(\frac{31r+t}{32}\right) \right\} + \frac{Y(\check{s}) - Y(r)}{(\check{s} - r)^2} \\
 & \times \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r+\check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+\check{s}}{2} \right)^3 \right\} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \Big| \\
 & \leq v(t)(\check{s} - r)(S - \gamma) \tag{2.3}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \frac{1}{4} \left[Y(t) + Y(r + \check{s} - t) + \frac{1}{2} \left\{ Y\left(\frac{r+t}{2}\right) + Y\left(\frac{r+2\check{s}-t}{2}\right) \right\} + \frac{1}{4} Y\left(\frac{3r+t}{4}\right) \right. \right. \\
 & \quad \left. \left. + Y\left(\frac{r+r-t}{4}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{7r+t}{8}\right) + Y\left(\frac{r+8\check{s}-t}{8}\right) \right\} + \frac{1}{16} \left\{ Y\left(\frac{15r+t}{16}\right) \right. \right. \\
 & \quad \left. \left. + Y\left(\frac{r+16\check{s}-t}{16}\right) + Y\left(\frac{31r+t}{32}\right) + Y\left(\frac{r+32\check{s}-t}{32}\right) \right\} + \left(t - \frac{5r+3\check{s}}{8} \right) \{ 2Y(r+\check{s}-t) \right. \right. \\
 & \quad \left. \left. - Y(t) \right\} + \frac{1}{2} \left\{ Y\left(\frac{r+2\check{s}-t}{2}\right) - Y\left(\frac{r+t}{2}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{r+4\check{s}-t}{4}\right) - Y\left(\frac{3r+t}{4}\right) \right\} \right. \\
 & \quad \left. + \frac{1}{32} \left\{ Y\left(\frac{r+8\check{s}-t}{8}\right) - Y\left(\frac{7r+t}{8}\right) \right\} + \frac{1}{128} \left\{ Y\left(\frac{r+16\check{s}-t}{16}\right) - Y\left(\frac{15r+t}{16}\right) \right\} \right\} \\
 & \quad + \frac{1}{512} \left(t - \frac{3r+\check{s}}{4} \right) \left\{ Y\left(\frac{r-32\check{s}-t}{32}\right) - Y\left(\frac{31r+t}{32}\right) \right\} + \frac{Y(\check{s}) - Y(r)}{(\check{s} - r)^2} \\
 & \times \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r+\check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+\check{s}}{2} \right)^3 \right\} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \Big| \\
 & \leq v(t)(\check{s} - r)(\Gamma - S), \tag{2.4}
 \end{aligned}$$

where

$$S = \frac{Y(\xi) - Y(r)}{\xi - r}$$

and

$$v(t) = \max \left\{ \left| \frac{1}{2048} (t - r)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{512} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \right. \\ \left| \frac{1}{128} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{32} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{8} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \\ \left| \frac{1}{2} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{2} \left(t - \frac{r + \xi}{2} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{8} \left(t - \frac{r + \xi}{2} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \\ \left| \frac{1}{32} \left(t - \frac{r + \xi}{2} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{128} \left(t - \frac{r + \xi}{2} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{512} \left(t - \frac{r + \xi}{2} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \\ \left. \left| \frac{1}{2048} \left(t - \frac{r + \xi}{2} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{\alpha(t)}{\xi - r} \right| \right\}.$$

Proof:

We know that

$$\frac{1}{\xi - r} \int_r^\xi Y''(\ddot{u}) d\ddot{u} - \frac{Y(\xi) - Y(r)}{\xi - r} \tag{2.5}$$

and

$$\frac{1}{\xi - r} \int_r^\xi P(t, \ddot{u}) d\ddot{u} = \frac{1}{\xi - r} \left[\frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \xi}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \xi}{2} \right)^3 \right], \tag{2.6}$$

which implies that

$$\frac{1}{\xi - r} \int_r^\xi P(t, \ddot{u}) Y''(\ddot{u}) d\ddot{u} - \frac{1}{(\xi - r)^2} \int_r^\xi P(t, \ddot{u}) \int_r^\xi Y''(\ddot{u}) d\ddot{u} \\ = \frac{1}{\xi - r} \int_r^\xi Y''(\ddot{u}) d\ddot{u} - \frac{1}{4} \left\{ Y(t) + Y(r + \xi - t) + \frac{1}{2} \left\{ Y\left(\frac{r + t}{2}\right) + Y\left(\frac{r + 2\xi - t}{2}\right) \right\} \right. \\ \left. + \frac{1}{4} \left\{ Y\left(\frac{3r + t}{4}\right) + Y\left(\frac{r + 4\xi - t}{4}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{7r + t}{8}\right) + Y\left(\frac{r + 8\xi - t}{8}\right) \right\} \right. \\ \left. + \frac{1}{16} \left\{ Y\left(\frac{15r + t}{16}\right) + Y\left(\frac{r + 16\xi - t}{16}\right) + Y\left(\frac{31r + t}{32}\right) + Y\left(\frac{r + 32\xi - t}{32}\right) \right\} \right. \\ \left. + \left(t - \frac{5t + 3\xi}{8} \right) \left\{ 2 \left\{ Y'(r + \xi - t) - Y'(t) \right\} + \frac{1}{2} \left\{ Y'\left(\frac{r + 2\xi - t}{2}\right) - Y'\left(\frac{r + t}{2}\right) \right\} \right. \right. \\ \left. \left. + \frac{1}{8} \left\{ Y'\left(\frac{r + 4\xi - t}{4}\right) - Y'\left(\frac{3r + t}{4}\right) \right\} + \frac{1}{32} \left\{ Y'\left(\frac{r + 8\xi - t}{8}\right) - Y'\left(\frac{7r + t}{8}\right) \right\} \right. \right. \\ \left. \left. + \frac{1}{128} \left\{ Y'\left(\frac{r + 16\xi - t}{16}\right) - Y'\left(\frac{15r + t}{16}\right) \right\} \right\} \right. \\ \left. + \frac{1}{512} \left(t - \frac{3r + \xi}{4} \right) \left\{ Y'\left(\frac{r + 32\xi - t}{32}\right) - Y'\left(\frac{31r + t}{32}\right) \right\} \right\} \\ - \frac{Y(\xi) - Y(r)}{(\xi - r)^2} \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \xi}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \xi}{2} \right)^3 \right\}. \tag{2.7}$$

We suppose that

$$R_n(t) = \frac{1}{\xi - r} \int_r^\xi P(t, \ddot{u}) Y''(\ddot{u}) d\ddot{u} - \frac{1}{(\xi - r)^2} \int_r^\xi P(t, \ddot{u}) \int_r^\xi Y''(\ddot{u}) d\ddot{u}. \tag{2.8}$$

If $C \in R$ is an arbitrary constant, then we have

$$R_n(t) = \frac{1}{\xi - r} \int_r^\xi (Y''(\ddot{u}) - c) \left[P(t, \ddot{u}) - \frac{1}{\xi - r} \int_r^\xi P(t, \xi) d\xi \right] d\ddot{u}. \tag{2.9}$$

Furthermore, we have

$$\leq \frac{1}{\xi - r} \max_{\ddot{u} \in [r, \xi]} \int_r^\xi (Y''(\ddot{u}) - c) \left| P(t, \ddot{u}) - \frac{1}{\xi - r} \int_r^\xi P(t, \xi) d\xi \right| \int_r^\xi |Y''(\ddot{u}) - c| d\ddot{u}. \tag{2.10}$$

Now

$$\max \left| P(t, \ddot{u}) - \frac{1}{\xi - r} \int_r^\xi P(t, \xi) d\xi \right| \\ = \max \left\{ \left| \frac{1}{2048} (t - r)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{512} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \right. \\ \left| \frac{1}{128} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{32} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \left| \frac{1}{8} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right|, \\ \left. \left| \frac{1}{2} \left(t - \frac{3r + \xi}{4} \right)^2 - \frac{\alpha(t)}{\xi - r} \right| \right\}$$

$$\left| \frac{1}{2} \left(t - \frac{3r + \check{s}}{4} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{2} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{8} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \\ \left| \frac{1}{32} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{128} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{512} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \\ \left. \left| \frac{1}{2048} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{\alpha(t)}{\check{s} - r} \right| \right\}. \tag{2.11}$$

Where

$$\alpha(t) = \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3.$$

We suppose that

$$v(t) = \max \left\{ \left| \frac{1}{2048} (t - r)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{512} \left(t - \frac{3r + \check{s}}{4} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \right. \\ \left. \left| \frac{1}{128} \left(t - \frac{3r + \check{s}}{4} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{32} \left(t - \frac{3r + \check{s}}{4} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{8} \left(t - \frac{3r + \check{s}}{4} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \right. \\ \left. \left| \frac{1}{2} \left(t - \frac{3r + \check{s}}{4} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{2} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{8} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \right. \\ \left. \left| \frac{1}{32} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{128} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{1}{512} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \right. \\ \left. \left| \frac{1}{2048} \left(t - \frac{r + \check{s}}{2} \right)^2 - \frac{\alpha(t)}{\check{s} - r} \right|, \left| \frac{\alpha(t)}{\check{s} - r} \right| \right\}. \tag{2.12}$$

We also have

$$\int_r^{\check{s}} |Y''(\check{u}) - \gamma| d\check{u} = (S - \gamma)(\check{s} - r), \tag{2.13}$$

$$\int_r^{\check{s}} |Y''(\check{u}) - \Gamma| d\check{u} = (\Gamma - S)(\check{s} - r). \tag{2.14}$$

So, we attain (2.3) and (2.4) by using (2.5)-(2.14) and taking $C = \gamma$ and $C = \Gamma$ in (2.10) respectively.

Corollary 1. By replacing $t = \frac{r + \check{s}}{2}$ (2.3) and (2.4), then

$$\left| \frac{1}{4} Y \left(\frac{r + \check{s}}{2} \right) + \frac{1}{8} \left\{ Y \left(\frac{3r + \check{s}}{4} \right) + Y \left(\frac{r + 3\check{s}}{4} \right) \right\} + \frac{1}{16} \left\{ Y \left(\frac{7r + \check{s}}{8} \right) + Y \left(\frac{r + 7\check{s}}{8} \right) \right\} \right. \\ \left. \frac{1}{32} \left\{ Y \left(\frac{15r + \check{s}}{16} \right) + Y \left(\frac{r + 15\check{s}}{16} \right) \right\} + \frac{1}{64} \left\{ Y \left(\frac{31r + \check{s}}{32} \right) + Y \left(\frac{r + 31\check{s}}{32} \right) + Y \left(\frac{63r + \check{s}}{64} \right) \right. \right. \\ \left. \left. + Y \left(\frac{r + 63\check{s}}{64} \right) \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ Y' \left(\frac{r + 3\check{s}}{4} \right) - Y' \left(\frac{3r + \check{s}}{4} \right) \right\} + \frac{1}{256} \left\{ Y' \left(\frac{r + 7\check{s}}{8} \right) \right. \right. \right. \\ \left. \left. - Y' \left(\frac{7r + \check{s}}{8} \right) \right\} + \frac{1}{1024} \left\{ Y' \left(\frac{r + 15\check{s}}{16} \right) - Y' \left(\frac{15r + \check{s}}{16} \right) \right\} + \frac{1}{4096} \left\{ Y' \left(\frac{r + 31\check{s}}{32} \right) \right. \right. \\ \left. \left. - Y' \left(\frac{31r + \check{s}}{32} \right) \right\} + \frac{1}{8192} \left\{ Y' \left(\frac{r + 63\check{s}}{64} \right) - Y' \left(\frac{63r + \check{s}}{64} \right) \right\} \right\} \\ \left. + \frac{2341}{393216} (\check{s} - r) \{ Y'(\check{s}) - Y'(r) \} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \right. \\ \left. \leq v \left(\frac{r + \check{s}}{2} \right) (S - \gamma) (\check{s} - r). \right.$$

And

$$\left| \frac{1}{4} Y \left(\frac{r + \check{s}}{2} \right) + \frac{1}{8} \left\{ Y \left(\frac{3r + \check{s}}{4} \right) + Y \left(\frac{r + 3\check{s}}{4} \right) \right\} + \frac{1}{16} \left\{ Y \left(\frac{7r + \check{s}}{8} \right) + Y \left(\frac{r + 7\check{s}}{8} \right) \right\} \right. \\ \left. \frac{1}{32} \left\{ Y \left(\frac{15r + \check{s}}{16} \right) + Y \left(\frac{r + 15\check{s}}{16} \right) \right\} + \frac{1}{64} \left\{ Y \left(\frac{31r + \check{s}}{32} \right) + Y \left(\frac{r + 31\check{s}}{32} \right) + Y \left(\frac{63r + \check{s}}{64} \right) \right. \right. \\ \left. \left. + Y \left(\frac{r + 63\check{s}}{64} \right) \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ Y' \left(\frac{r + 3\check{s}}{4} \right) - Y' \left(\frac{3r + \check{s}}{4} \right) \right\} + \frac{1}{256} \left\{ Y' \left(\frac{r + 7\check{s}}{8} \right) \right. \right. \right. \\ \left. \left. - Y' \left(\frac{7r + \check{s}}{8} \right) \right\} + \frac{1}{1024} \left\{ Y' \left(\frac{r + 15\check{s}}{16} \right) - Y' \left(\frac{15r + \check{s}}{16} \right) \right\} + \frac{1}{4096} \left\{ Y' \left(\frac{r + 31\check{s}}{32} \right) \right. \right. \\ \left. \left. - Y' \left(\frac{31r + \check{s}}{32} \right) \right\} + \frac{1}{8192} \left\{ Y' \left(\frac{r + 63\check{s}}{64} \right) - Y' \left(\frac{63r + \check{s}}{64} \right) \right\} \right\} \\ \left. + \frac{2341}{393216} (\check{s} - r) \{ Y'(\check{s}) - Y'(r) \} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \right.$$

$$\leq v \left(\frac{r + \check{s}}{2} \right) (\Gamma - S)(\check{s} - r).$$

Case 2: For $Y''' \in L^2[r, \check{s}]$

Theorem 2. Let $Y: [r, \check{s}] \rightarrow \mathbb{R}$ be three times differentiable function on (r, \check{s}) . If $Y''' \in L^2[r, \check{s}]$, then for all $t \in [r, \frac{r+\check{s}}{2}]$, we have

$$\begin{aligned} & \left| \frac{1}{4} \left[Y(t) + Y(r + \check{s} - t) + \frac{1}{2} \left\{ Y\left(\frac{r+t}{2}\right) + Y\left(\frac{r+2\check{s}-t}{2}\right) \right\} + \frac{1}{4} \left\{ Y\left(\frac{3r+t}{4}\right) \right. \right. \right. \\ & \left. \left. + Y\left(\frac{r+4\check{s}-t}{4}\right) \right\} + \frac{1}{8} \left\{ Y\left(\frac{7r+t}{8}\right) + Y\left(\frac{r+8\check{s}-t}{8}\right) \right\} + \frac{1}{16} \left\{ Y\left(\frac{15r+t}{16}\right) \right. \right. \\ & \left. \left. + Y\left(\frac{r+16\check{s}-t}{16}\right) + Y\left(\frac{31r+t}{32}\right) + Y\left(\frac{r+32\check{s}-t}{32}\right) \right\} + \left(t - \frac{5r+3\check{s}}{8} \right) \right. \\ & \left. \times \left\{ 2 \{ Y'(r + \check{s} - t) - Y'(t) \} + \frac{1}{2} \left\{ Y'\left(\frac{r+2\check{s}-t}{2}\right) - Y'\left(\frac{r+t}{2}\right) \right\} \right. \right. \\ & \left. \left. + \frac{1}{8} \left\{ Y'\left(\frac{r+4\check{s}-t}{4}\right) - Y'\left(\frac{3r+t}{4}\right) \right\} + \frac{1}{32} \left\{ Y'\left(\frac{r+8\check{s}-t}{8}\right) - Y'\left(\frac{7r+t}{8}\right) \right\} \right. \right. \\ & \left. \left. + \frac{1}{128} \left\{ Y'\left(\frac{r+16\check{s}-t}{16}\right) - Y'\left(\frac{15r+t}{16}\right) \right\} \right\} + \frac{1}{512} \left(t - \frac{3r+\check{s}}{4} \right) \right. \\ & \left. \times \left\{ Y'\left(\frac{r+32\check{s}-t}{32}\right) - Y'\left(\frac{31r+t}{32}\right) \right\} \right] + \frac{Y'(\check{s}) - Y'(r)}{(\check{s} - r)^2} \left\{ \frac{1}{98304} (t - r)^3 \right. \right. \\ & \left. \left. + \frac{4681}{12288} \left(t - \frac{3r+\check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+\check{s}}{2} \right)^3 \right\} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \right| \\ & \leq \frac{1}{\pi} \|Y'''\|_2 \left[\frac{1}{167772160} (t - r)^5 + \frac{1082401}{5242880} \left(t - \frac{3r+\check{s}}{4} \right)^5 \right. \\ & \left. - \frac{34636833}{167772160} \left(t - \frac{r+\check{s}}{2} \right)^5 \right. \\ & \left. - \frac{1}{\check{s} - r} \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r+\check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+\check{s}}{2} \right)^3 \right\}^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{2.15}$$

Proof: Let $R_n(t)$ is defined as in (2.8)

$R_n(t)$

$$= \frac{1}{\check{s} - r} \int_r^{\check{s}} P(t, \check{u}) Y''(\check{u}) d\check{u} - \frac{1}{(\check{s} - r)^2} \int_r^{\check{s}} P(t, \check{u}) \int_r^{\check{s}} Y''(\check{u}) d\check{u}.$$

If we take $C = Y''\left(\frac{r+\check{s}}{2}\right)$ in (2.9) and by applying the Cauchy inequality, then

$$\begin{aligned} |R_n(t)| & \leq \frac{1}{\check{s} - r} \int_r^{\check{s}} \left| Y''(\check{u}) - Y''\left(\frac{r+\check{s}}{2}\right) \right| \left| P(t, \check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} P(t, \check{s}) d\check{s} \right| d\check{u} \\ & \leq \frac{1}{\check{s} - r} \left[\int_r^{\check{s}} \left(Y''(\check{u}) - Y''\left(\frac{r+\check{s}}{2}\right) \right)^2 d\check{u} \right]^{\frac{1}{2}} \left[\int_r^{\check{s}} \left(P(t, \check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} P(t, \check{s}) d\check{s} \right)^2 d\check{u} \right]^{\frac{1}{2}}. \end{aligned}$$

We apply the Diaz-Metcalf inequality to get

$$\int_r^{\check{s}} \left(Y''(\check{u}) - Y''\left(\frac{r+\check{s}}{2}\right) \right)^2 d\check{u} \leq \frac{(\check{s} - r)^2}{\pi^2} \|Y'''\|_2^2.$$

So

$$\begin{aligned} & \int_r^{\check{s}} \left(P(t, \check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} P(t, \check{s}) d\check{s} \right)^2 d\check{u} \\ & = \frac{1}{167772160} (t - r)^5 + \frac{1082401}{5242880} \left(t - \frac{3r+\check{s}}{4} \right)^5 - \frac{34636833}{167772160} \left(t - \frac{r+\check{s}}{2} \right)^5 \\ & \quad - \frac{1}{\check{s} - r} \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r+\check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+\check{s}}{2} \right)^3 \right\}^2. \end{aligned} \tag{2.16}$$

By using (2.7) in above inequalities, we get (2.15).

Corollary 2. By replacing $t = \frac{r+\check{s}}{2}$ in (2.15), we get

$$\begin{aligned} & \left| \frac{1}{4} Y\left(\frac{r+\check{s}}{2}\right) + \frac{1}{8} \left\{ Y\left(\frac{3r+\check{s}}{4}\right) + Y\left(\frac{r+3\check{s}}{4}\right) \right\} + \frac{1}{16} \left\{ Y\left(\frac{7r+\check{s}}{8}\right) + Y\left(\frac{r+7\check{s}}{8}\right) \right\} \right. \\ & \left. \frac{1}{32} \left\{ Y\left(\frac{15r+\check{s}}{16}\right) + Y\left(\frac{r+15\check{s}}{16}\right) \right\} + \frac{1}{64} \left\{ Y\left(\frac{31r+\check{s}}{32}\right) + Y\left(\frac{r+31\check{s}}{32}\right) + Y\left(\frac{63r+\check{s}}{64}\right) \right. \right. \\ & \left. \left. + Y\left(\frac{r+63\check{s}}{64}\right) \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ Y'\left(\frac{r+3\check{s}}{4}\right) - Y'\left(\frac{3r+\check{s}}{4}\right) \right\} + \frac{1}{256} \left\{ Y'\left(\frac{r+7\check{s}}{8}\right) \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
 & -Y' \left(\frac{7r + \check{s}}{8} \right) \Big\} + \frac{1}{1024} \left\{ Y' \left(\frac{r + 15\check{s}}{16} \right) - Y' \left(\frac{15r + \check{s}}{16} \right) \right\} + \frac{1}{4096} \left\{ Y' \left(\frac{r + 31\check{s}}{32} \right) \right. \\
 & \left. - Y' \left(\frac{31r + \check{s}}{32} \right) \right\} + \frac{1}{8192} \left\{ Y' \left(\frac{r + 63\check{s}}{64} \right) - Y' \left(\frac{63r + \check{s}}{64} \right) \right\} \\
 & + \frac{2341}{393216} (\check{s} - r) \{ Y'(\check{s}) - Y'(r) \} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \\
 & \leq \frac{1}{\pi} \|Y'''\|_2 (\check{s} - r)^{\frac{5}{2}} \frac{1}{196608} \sqrt{\frac{2337080251}{10}}.
 \end{aligned}$$

Case 3: For $Y'' \in L^2[r, \check{s}]$

Theorem 3. Let $Y: [r, \check{s}] \rightarrow \mathbb{R}$ be an absolutely continuous function on (r, \check{s}) , with $Y''' \in L^2[r, \check{s}]$. Then

$$\begin{aligned}
 & \left| \frac{1}{4} \left[Y(t) + Y(r + \check{s} - t) + \frac{1}{2} \left\{ Y \left(\frac{r + t}{2} \right) + Y \left(\frac{r + 2\check{s} - t}{2} \right) \right\} + \frac{1}{4} \left\{ Y \left(\frac{3r + t}{4} \right) \right. \right. \right. \\
 & \left. \left. + Y \left(\frac{r + 4\check{s} - t}{4} \right) \right\} + \frac{1}{8} \left\{ Y \left(\frac{7r + t}{8} \right) + Y \left(\frac{r + 8\check{s} - t}{8} \right) \right\} + \frac{1}{16} \left\{ Y \left(\frac{15r + t}{16} \right) \right. \right. \\
 & \left. \left. + Y \left(\frac{r + 16\check{s} - t}{16} \right) + Y \left(\frac{31r + t}{32} \right) + Y \left(\frac{r + 32\check{s} - t}{32} \right) \right\} + \left(t - \frac{5r + 3\check{s}}{8} \right) \right. \\
 & \left. \times \left\{ 2 \{ Y'(r + \check{s} - t) - Y'(t) \} + \frac{1}{2} \left\{ Y' \left(\frac{r + 2\check{s} - t}{2} \right) - Y' \left(\frac{r + t}{2} \right) \right\} \right. \right. \\
 & \left. \left. + \frac{1}{8} \left\{ Y' \left(\frac{r + 4\check{s} - t}{4} \right) - Y' \left(\frac{3r + t}{4} \right) \right\} + \frac{1}{32} \left\{ Y' \left(\frac{r + 8\check{s} - t}{8} \right) - Y' \left(\frac{7r + t}{8} \right) \right\} \right. \right. \\
 & \left. \left. + \frac{1}{128} \left\{ Y' \left(\frac{r + 16\check{s} - t}{16} \right) - Y' \left(\frac{15r + t}{16} \right) \right\} \right\} + \frac{1}{512} \left(t - \frac{3r + \check{s}}{4} \right) \right. \\
 & \left. \times \left\{ Y' \left(\frac{r + 32\check{s} - t}{32} \right) - Y' \left(\frac{31r + t}{32} \right) \right\} \right] + \frac{Y'(\check{s}) - Y'(r)}{(\check{s} - r)^2} \left\{ \frac{1}{98304} (t - r)^3 \right. \right. \\
 & \left. \left. + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \right\} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \right| \\
 & \leq \frac{\sqrt{\sigma(Y'')}}{\check{s} - r} \left[\frac{1}{167772160} (t - r)^5 + \frac{1082401}{5242880} \left(t - \frac{3r + \check{s}}{4} \right)^5 - \frac{34636833}{167772160} \left(t - \frac{r + \check{s}}{2} \right)^5 \right. \\
 & \left. - \frac{1}{\check{s} - r} \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \right\}^2 \right]^{\frac{1}{2}}, \tag{2.17}
 \end{aligned}$$

$\forall t \in \left[r, \frac{r + \check{s}}{2} \right]$,

where

$$\sigma(Y'') = \|Y''\|_2 - \frac{(Y'(\check{s}) - Y'(r))^2}{\check{s} - r} = \|Y''\|_2^2 - \check{s}^2 (\check{s} - r)$$

and S is defined in Theorem 1.

Proof: Let $R_n(t)$ be defined as in (2.8). If we take $C = \frac{1}{\check{s} - r} \int_r^{\check{s}} Y''(\check{s}) d\check{s}$ in (2.9) and by applying the Cauchy inequality and (2.16), then

$$\begin{aligned}
 & |R_n(t)| \leq \frac{1}{\check{s} - r} \int_r^{\check{s}} \left| Y''(\check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y''(\check{s}) d\check{s} \right| \left| P(t, \check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} P(t, \check{s}) d\check{s} \right| d\check{u} \\
 & \leq \frac{1}{\check{s} - r} \left[\int_r^{\check{s}} \left(Y''(\check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y''(\check{s}) d\check{s} \right)^2 d\check{u} \right]^{\frac{1}{2}} \left[\int_r^{\check{s}} \left(P(t, \check{u}) - \frac{1}{\check{s} - r} \int_r^{\check{s}} P(t, \check{s}) d\check{s} \right)^2 d\check{u} \right]^{\frac{1}{2}} \\
 & = \frac{\sqrt{\sigma(Y'')}}{\check{s} - r} \left[\frac{1}{167772160} (t - r)^5 + \frac{1082401}{5242880} \left(t - \frac{3r + \check{s}}{4} \right)^5 - \frac{34636833}{167772160} \left(t - \frac{r + \check{s}}{2} \right)^5 \right. \\
 & \left. - \frac{1}{\check{s} - r} \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \right\}^2 \right]^{\frac{1}{2}}.
 \end{aligned}$$

By using above equations, we get (2.17).

Corollary 3. By replacing $t = \frac{r + \check{s}}{2}$ in (2.17), we get

$$\begin{aligned}
 & \left| \frac{1}{4} Y \left(\frac{r + \check{s}}{2} \right) + \frac{1}{8} \left\{ Y \left(\frac{3r + \check{s}}{4} \right) + Y \left(\frac{r + 3\check{s}}{4} \right) \right\} + \frac{1}{16} \left\{ Y \left(\frac{7r + \check{s}}{8} \right) + Y \left(\frac{r + 7\check{s}}{8} \right) \right\} \right. \\
 & \left. \frac{1}{32} \left\{ Y \left(\frac{15r + \check{s}}{16} \right) + Y \left(\frac{r + 15\check{s}}{16} \right) \right\} + \frac{1}{64} \left\{ Y \left(\frac{31r + \check{s}}{32} \right) + Y \left(\frac{r + 31\check{s}}{32} \right) + Y \left(\frac{63r + \check{s}}{64} \right) \right. \right. \\
 & \left. \left. + Y \left(\frac{r + 63\check{s}}{64} \right) \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ Y' \left(\frac{r + 3\check{s}}{4} \right) - Y' \left(\frac{3r + \check{s}}{4} \right) \right\} + \frac{1}{256} \left\{ Y' \left(\frac{r + 7\check{s}}{8} \right) \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & -Y' \left(\frac{7r + \check{s}}{8} \right) \Big\} + \frac{1}{1024} \left\{ Y' \left(\frac{r + 15\check{s}}{16} \right) - Y' \left(\frac{15r + \check{s}}{16} \right) \right\} + \frac{1}{4096} \left\{ Y' \left(\frac{r + 31\check{s}}{32} \right) \right. \\
 & \left. - Y' \left(\frac{31r + \check{s}}{32} \right) \right\} + \frac{1}{8192} \left\{ Y' \left(\frac{r + 63\check{s}}{64} \right) - Y' \left(\frac{63r + \check{s}}{64} \right) \right\} \\
 & + \frac{2341}{393216} (\check{s} - r) \{ Y'(\check{s}) - Y'(r) \} - \frac{1}{\check{s} - r} \int_r^{\check{s}} Y(\check{u}) d\check{u} \\
 & \leq \sqrt{\sigma(Y'')} (\check{s} - r)^{\frac{5}{2}} \frac{1}{196608} \sqrt{\frac{2337080251}{10}}.
 \end{aligned}$$

3. An Applications to Cumulative Distributive Function

Let X be a random variable taking values in the finite interval [r, s] with the probability density function Y: [r, s] → [0,1] and cumulative distributive function

$$\begin{aligned}
 F(t) &= Pr(T \leq t) = \int_r^t Y(\check{u}) d\check{u}, \\
 F(\check{s}) &= Pr(T \leq \check{s}) = \int_r^{\check{s}} Y(e) de = 1.
 \end{aligned}$$

Theorem 4. With the assumptions of Theorem 1, we have the following inequality which holds

$$\begin{aligned}
 & \left| \frac{\check{s} - E(T)}{\check{s} - r} - \frac{1}{4} \left[F(t) + F(r + \check{s} - t) + \frac{1}{2} \left\{ F \left(\frac{r + t}{2} \right) + F \left(\frac{r + 2\check{s} - t}{2} \right) \right\} + \frac{1}{4} \left\{ F \left(\frac{3r + t}{4} \right) \right. \right. \right. \\
 & \left. \left. + F \left(\frac{r + 4\check{s} - t}{4} \right) \right\} + \frac{1}{8} \left\{ F \left(\frac{7r + t}{8} \right) + F \left(\frac{r + 8\check{s} - t}{8} \right) \right\} + \frac{1}{16} \left\{ F \left(\frac{15r + t}{16} \right) \right. \right. \\
 & \left. \left. + F \left(\frac{r + 16\check{s} - t}{16} \right) + F \left(\frac{31r + t}{32} \right) + F \left(\frac{r + 32\check{s} - t}{32} \right) \right\} + \left(t - \frac{5r + 3\check{s}}{8} \right) \{ 2\{ F'(r + \check{s} - t) \right. \right. \\
 & \left. \left. - F'(t) \} + \frac{1}{2} \left\{ F' \left(\frac{r + 2\check{s} - t}{2} \right) - F' \left(\frac{r + t}{2} \right) \right\} + \frac{1}{8} \left\{ F' \left(\frac{r + 4\check{s} - t}{4} \right) - F' \left(\frac{3r + t}{4} \right) \right\} \right. \\
 & \left. \left. + \frac{1}{32} \left\{ F' \left(\frac{r + 8\check{s} - t}{8} \right) - F' \left(\frac{7r + t}{8} \right) \right\} + \frac{1}{128} \left\{ F' \left(\frac{r + 16\check{s} - t}{16} \right) - F' \left(\frac{15r + t}{16} \right) \right\} \right\} \\
 & \left. + \frac{1}{512} \left(t - \frac{3r + \check{s}}{4} \right) \left\{ F' \left(\frac{r + 32\check{s} - t}{32} \right) - F' \left(\frac{31r + t}{32} \right) \right\} \right] - \frac{F'(\check{s}) - F'(r)}{(\check{s} - r)^2} \\
 & \times \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \Big| \\
 & \leq v(t)(\check{s} - r)(S - \gamma)
 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
 & \left| \frac{\check{s} - E(T)}{\check{s} - r} - \frac{1}{4} \left[F(t) + F(r + \check{s} - t) + \frac{1}{2} \left\{ F \left(\frac{r + t}{2} \right) + F \left(\frac{r + 2\check{s} - t}{2} \right) \right\} + \frac{1}{4} \left\{ F \left(\frac{3r + t}{4} \right) \right. \right. \right. \\
 & \left. \left. + F \left(\frac{r + 4\check{s} - t}{4} \right) \right\} + \frac{1}{8} \left\{ F \left(\frac{7r + t}{8} \right) + F \left(\frac{r + 8\check{s} - t}{8} \right) \right\} + \frac{1}{16} \left\{ F \left(\frac{15r + t}{16} \right) \right. \right. \\
 & \left. \left. + F \left(\frac{r + 16\check{s} - t}{16} \right) + F \left(\frac{31r + t}{32} \right) + F \left(\frac{r + 32\check{s} - t}{32} \right) \right\} + \left(t - \frac{5r + 3\check{s}}{8} \right) \{ 2\{ F'(r + \check{s} - t) \right. \right. \\
 & \left. \left. - F'(t) \} + \frac{1}{2} \left\{ F' \left(\frac{r + 2\check{s} - t}{2} \right) - F' \left(\frac{r + t}{2} \right) \right\} + \frac{1}{8} \left\{ F' \left(\frac{r + 4\check{s} - t}{4} \right) - F' \left(\frac{3r + t}{4} \right) \right\} \right. \\
 & \left. \left. + \frac{1}{32} \left\{ F' \left(\frac{r + 8\check{s} - t}{8} \right) - F' \left(\frac{7r + t}{8} \right) \right\} + \frac{1}{128} \left\{ F' \left(\frac{r + 16\check{s} - t}{16} \right) - F' \left(\frac{15r + t}{16} \right) \right\} \right\} \\
 & \left. + \frac{1}{512} \left(t - \frac{3r + \check{s}}{4} \right) \left\{ F' \left(\frac{r + 32\check{s} - t}{32} \right) - F' \left(\frac{31r + t}{32} \right) \right\} \right] - \frac{F'(\check{s}) - F'(r)}{(\check{s} - r)^2} \\
 & \times \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \right\} \Big| \\
 & \leq v(t)(\check{s} - r)(\Gamma - S).
 \end{aligned} \tag{3.2}$$

∀ t ∈ [r, $\frac{r+\check{s}}{2}$], where E(T) is the expectation of T.

Proof: By (2.3) and (2.4) on choosing Y = F and using the fact

$$E(T) = \int_r^t \check{u} dF(U) = t - \int_r^t F(\check{u}) d\check{u},$$

we obtain (3.1) and (3.2).

Corollary 4. By replacing t = $\frac{r+\check{s}}{2}$ in (3.1) and (3.2), then

$$\begin{aligned} & \left| \frac{\check{s} - E(T)}{\check{s} - r} - \left[\frac{1}{2} F\left(\frac{r + \check{s}}{2}\right) + \frac{1}{8} \left\{ F\left(\frac{3r + \check{s}}{4}\right) + F\left(\frac{r + 3\check{s}}{4}\right) \right\} + \frac{1}{16} \left\{ F\left(\frac{7r + \check{s}}{8}\right) \right. \right. \\ & + F\left(\frac{r + 7\check{s}}{8}\right) \left. \right\} + \frac{1}{32} \left\{ F\left(\frac{15r + \check{s}}{16}\right) + F\left(\frac{r + 15\check{s}}{16}\right) \right\} + \frac{1}{64} \left\{ F\left(\frac{31r + \check{s}}{32}\right) + F\left(\frac{r + 31\check{s}}{32}\right) \right. \\ & + F\left(\frac{63r + \check{s}}{64}\right) + F\left(\frac{r + 63\check{s}}{64}\right) \left. \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ F'\left(\frac{r + 3\check{s}}{4}\right) - F'\left(\frac{3r + \check{s}}{4}\right) \right\} \right. \\ & + \frac{1}{256} \left\{ F'\left(\frac{r + 7\check{s}}{8}\right) - F'\left(\frac{7r + \check{s}}{8}\right) \right\} + \frac{1}{1024} \left\{ F'\left(\frac{r + 15\check{s}}{16}\right) - F'\left(\frac{15r + \check{s}}{16}\right) \right\} \\ & + \frac{1}{4096} \left\{ F'\left(\frac{r + 31\check{s}}{32}\right) - F'\left(\frac{31r + \check{s}}{32}\right) \right\} + \frac{1}{8192} \left\{ F'\left(\frac{r + 63\check{s}}{64}\right) - F'\left(\frac{63r + \check{s}}{64}\right) \right\} \left. \right\} \\ & - \frac{2341}{393216} (\check{s} - r) \{ F'(\check{s}) - F'(r) \} \Big| \\ & \leq v(t) (\check{s} - r) (S - \gamma) \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{\check{s} - E(T)}{\check{s} - r} - \left[\frac{1}{2} F\left(\frac{r + \check{s}}{2}\right) + \frac{1}{8} \left\{ F\left(\frac{3r + \check{s}}{4}\right) + F\left(\frac{r + 3\check{s}}{4}\right) \right\} + \frac{1}{16} \left\{ F\left(\frac{7r + \check{s}}{8}\right) \right. \right. \\ & + F\left(\frac{r + 7\check{s}}{8}\right) \left. \right\} + \frac{1}{32} \left\{ F\left(\frac{15r + \check{s}}{16}\right) + F\left(\frac{r + 15\check{s}}{16}\right) \right\} + \frac{1}{64} \left\{ F\left(\frac{31r + \check{s}}{32}\right) + F\left(\frac{r + 31\check{s}}{32}\right) \right. \\ & + F\left(\frac{63r + \check{s}}{64}\right) + F\left(\frac{r + 63\check{s}}{64}\right) \left. \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ F'\left(\frac{r + 3\check{s}}{4}\right) - F'\left(\frac{3r + \check{s}}{4}\right) \right\} \right. \\ & + \frac{1}{256} \left\{ F'\left(\frac{r + 7\check{s}}{8}\right) - F'\left(\frac{7r + \check{s}}{8}\right) \right\} + \frac{1}{1024} \left\{ F'\left(\frac{r + 15\check{s}}{16}\right) - F'\left(\frac{15r + \check{s}}{16}\right) \right\} \\ & + \frac{1}{4096} \left\{ F'\left(\frac{r + 31\check{s}}{32}\right) - F'\left(\frac{31r + \check{s}}{32}\right) \right\} + \frac{1}{8192} \left\{ F'\left(\frac{r + 63\check{s}}{64}\right) - F'\left(\frac{63r + \check{s}}{64}\right) \right\} \left. \right\} \\ & - \frac{2341}{393216} (\check{s} - r) \{ F'(\check{s}) - F'(r) \} \Big| \\ & \leq v\left(\frac{r + \check{s}}{2}\right) (\Gamma - S) (\check{s} - r). \end{aligned}$$

Theorem 5. With the assumptions of Theorem 2, we have the following inequality which holds

$$\begin{aligned} & \left| \frac{\check{s} - E(T)}{\check{s} - r} - \frac{1}{4} \left[F(t) + F(r + \check{s} - t) + \frac{1}{2} \left\{ F\left(\frac{r + t}{2}\right) + F\left(\frac{r + 2\check{s} - t}{2}\right) \right\} + \frac{1}{4} \left\{ F\left(\frac{3r + t}{4}\right) \right. \right. \right. \\ & + F\left(\frac{r + 4\check{s} - t}{4}\right) \left. \right\} + \frac{1}{8} \left\{ F\left(\frac{7r + t}{8}\right) + F\left(\frac{r + 8\check{s} - t}{8}\right) \right\} + \frac{1}{16} \left\{ F\left(\frac{15r + t}{16}\right) \right. \\ & + F\left(\frac{r + 16\check{s} - t}{16}\right) + F\left(\frac{31r + t}{32}\right) + F\left(\frac{r + 32\check{s} - t}{32}\right) \left. \right\} + \left(t - \frac{5r + 3\check{s}}{8} \right) \left\{ 2 \{ F'(r + \check{s} - t) \right. \right. \\ & - F'(t) \} + \frac{1}{2} \left\{ F'\left(\frac{r + 2\check{s} - t}{2}\right) - F'\left(\frac{r + t}{2}\right) \right\} + \frac{1}{8} \left\{ F'\left(\frac{r + 4\check{s} - t}{4}\right) - F'\left(\frac{3r + t}{4}\right) \right\} \\ & + \frac{1}{32} \left\{ F'\left(\frac{r + 8\check{s} - t}{8}\right) - F'\left(\frac{7r + t}{8}\right) \right\} + \frac{1}{128} \left\{ F'\left(\frac{r + 16\check{s} - t}{16}\right) - F'\left(\frac{15r + t}{16}\right) \right\} \left. \right\} \\ & + \frac{1}{512} \left(t - \frac{3r + \check{s}}{4} \right) \left\{ F'\left(\frac{r + 32\check{s} - t}{32}\right) - F'\left(\frac{31r + t}{32}\right) \right\} - \frac{F'(\check{s}) - F'(r)}{(\check{s} - r)^2} \\ & \times \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \Big| \\ & \leq \frac{1}{\pi} \|F'''\|_2 \left[\frac{1}{167772160} (t - r)^5 + \frac{1082401}{5242880} \left(t - \frac{3r + \check{s}}{4} \right)^5 - \frac{34636833}{167772160} \left(t - \frac{r + \check{s}}{2} \right)^5 \right. \\ & \left. - \frac{1}{\check{s} - r} \left\{ \frac{1}{98304} (t - r)^3 + \frac{4681}{12288} \left(t - \frac{3r + \check{s}}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r + \check{s}}{2} \right)^3 \right\}^2 \right]^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

$\forall t \in \left[r, \frac{r + \check{s}}{2} \right]$, where $E(T)$ is the expectation of T .

Proof: By using (2.15) and the same condition that we use in above theorem, we get the required inequality (3.3).

Corollary 5. By replacing $t = \frac{r + \check{s}}{2}$ in (3.3), then

$$\begin{aligned} & \left| \frac{\check{s} - E(T)}{\check{s} - r} - \left[\frac{1}{2} F\left(\frac{r + \check{s}}{2}\right) + \frac{1}{8} \left\{ F\left(\frac{3r + \check{s}}{4}\right) + F\left(\frac{r + 3\check{s}}{4}\right) \right\} + \frac{1}{16} \left\{ F\left(\frac{7r + \check{s}}{8}\right) \right. \right. \\ & + F\left(\frac{r + 7\check{s}}{8}\right) \left. \right\} + \frac{1}{32} \left\{ F\left(\frac{15r + \check{s}}{16}\right) + F\left(\frac{r + 15\check{s}}{16}\right) \right\} + \frac{1}{64} \left\{ F\left(\frac{31r + \check{s}}{32}\right) + F\left(\frac{r + 31\check{s}}{32}\right) \right. \\ & + F\left(\frac{63r + \check{s}}{64}\right) + F\left(\frac{r + 63\check{s}}{64}\right) \left. \right\} + (\check{s} - r) \left\{ \frac{1}{64} \left\{ F'\left(\frac{r + 3\check{s}}{4}\right) - F'\left(\frac{3r + \check{s}}{4}\right) \right\} \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{256} \left\{ F' \left(\frac{r+7s}{8} \right) - F' \left(\frac{7r+s}{8} \right) \right\} + \frac{1}{1024} \left\{ F' \left(\frac{r+15s}{16} \right) - F' \left(\frac{15r+s}{16} \right) \right\} \\
 & + \frac{1}{4096} \left\{ F' \left(\frac{r+31s}{32} \right) - F' \left(\frac{31r+s}{32} \right) \right\} + \frac{1}{8192} \left\{ F' \left(\frac{r+63s}{64} \right) - F' \left(\frac{63r+s}{64} \right) \right\} \\
 & - \frac{2341}{393216} (s-r) \left| \{ F'(s) - F'(r) \} \right| \\
 & \leq \frac{1}{\pi} \|F'''\|_2 (s-r)^{\frac{5}{2}} \frac{1}{196608} \sqrt{\frac{2337080251}{10}}.
 \end{aligned}$$

Theorem 6. With the assumptions of *Theorem 3*, we have the following inequality which holds

$$\begin{aligned}
 & \left| \frac{s-E(T)}{s-r} - \frac{1}{4} \left[F(t) + F(r+s-t) + \frac{1}{2} \left\{ F \left(\frac{r+t}{2} \right) + F \left(\frac{r+2s-t}{2} \right) \right\} + \frac{1}{4} \left\{ F \left(\frac{3r+t}{4} \right) \right. \right. \right. \\
 & \quad \left. \left. + F \left(\frac{r+4s-t}{4} \right) \right\} + \frac{1}{8} \left\{ F \left(\frac{7r+t}{8} \right) + F \left(\frac{r+8s-t}{8} \right) \right\} + \frac{1}{16} \left\{ F \left(\frac{15r+t}{16} \right) \right. \right. \\
 & \quad \left. \left. + F \left(\frac{r+16s-t}{16} \right) + F \left(\frac{31r+t}{32} \right) + F \left(\frac{r+32s-t}{32} \right) \right\} + \left(t - \frac{5r+3s}{8} \right) \{ 2 \{ F'(r+s-t) \right. \right. \\
 & \quad \left. \left. - F'(t) \} + \frac{1}{2} \left\{ F' \left(\frac{r+2s-t}{2} \right) - F' \left(\frac{r+t}{2} \right) \right\} + \frac{1}{8} \left\{ F' \left(\frac{r+4s-t}{4} \right) - F' \left(\frac{3r+t}{4} \right) \right\} \right. \\
 & \quad \left. \left. + \frac{1}{32} \left\{ F' \left(\frac{r+8s-t}{8} \right) - F' \left(\frac{7r+t}{8} \right) \right\} + \frac{1}{128} \left\{ F' \left(\frac{r+16s-t}{16} \right) - F' \left(\frac{15r+t}{16} \right) \right\} \right\} \right. \\
 & \quad \left. + \frac{1}{512} \left(t - \frac{3r+s}{4} \right) \left\{ F' \left(\frac{r+32s-t}{32} \right) - F' \left(\frac{31r+t}{32} \right) \right\} - \frac{F'(s) - F'(r)}{(s-r)^2} \right. \\
 & \quad \left. \times \left\{ \frac{1}{98304} (t-r)^3 + \frac{4681}{12288} \left(t - \frac{3r+s}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+s}{2} \right)^3 \right\} \right| \\
 & \leq \frac{\sqrt{\sigma(F'')}}{s-r} \left[\frac{1}{167772160} (t-r)^5 + \frac{1082401}{5242880} \left(t - \frac{3r+s}{4} \right)^5 - \frac{34636833}{167772160} \left(t - \frac{r+s}{2} \right)^5 \right. \\
 & \quad \left. - \frac{1}{s-r} \left\{ \frac{1}{98304} (t-r)^3 + \frac{4681}{12288} \left(t - \frac{3r+s}{4} \right)^3 - \frac{12483}{32768} \left(t - \frac{r+s}{2} \right)^3 \right\}^2 \right]^{\frac{1}{2}}. \tag{3.4}
 \end{aligned}$$

$\forall t \in \left[r, \frac{r+s}{2} \right]$, where $E(T)$ is the expectation of T .

Proof: By using (2.17) and the same condition that we use in above theorem, we get the required inequality (3.4).

Corollary 6. By replacing $t = \frac{r+s}{2}$ in (3.4), then

$$\begin{aligned}
 & \left| \frac{s-E(T)}{s-r} - \left[\frac{1}{2} F \left(\frac{r+s}{2} \right) + \frac{1}{8} \left\{ F \left(\frac{3r+s}{4} \right) + F \left(\frac{r+3s}{4} \right) \right\} + \frac{1}{16} \left\{ F \left(\frac{7r+s}{8} \right) \right. \right. \right. \\
 & \quad \left. \left. + F \left(\frac{r+7s}{8} \right) \right\} + \frac{1}{32} \left\{ F \left(\frac{15r+s}{16} \right) + F \left(\frac{r+15s}{16} \right) \right\} + \frac{1}{64} \left\{ F \left(\frac{31r+s}{32} \right) + F \left(\frac{r+31s}{32} \right) \right. \right. \\
 & \quad \left. \left. + F \left(\frac{63r+s}{64} \right) + F \left(\frac{r+63s}{64} \right) \right\} + (s-r) \left\{ \frac{1}{64} \left\{ F' \left(\frac{r+3s}{4} \right) - F' \left(\frac{3r+s}{4} \right) \right\} \right. \right. \\
 & \quad \left. \left. + \frac{1}{256} \left\{ F' \left(\frac{r+7s}{8} \right) - F' \left(\frac{7r+s}{8} \right) \right\} + \frac{1}{1024} \left\{ F' \left(\frac{r+15s}{16} \right) - F' \left(\frac{15r+s}{16} \right) \right\} \right. \right. \\
 & \quad \left. \left. + \frac{1}{4096} \left\{ F' \left(\frac{r+31s}{32} \right) - F' \left(\frac{31r+s}{32} \right) \right\} + \frac{1}{8192} \left\{ F' \left(\frac{r+63s}{64} \right) - F' \left(\frac{63r+s}{64} \right) \right\} \right\} \right. \\
 & \quad \left. - \frac{2341}{393216} (s-r) \left| \{ F'(s) - F'(r) \} \right| \right| \\
 & \leq \sqrt{\sigma(F'')} (s-r)^{\frac{3}{2}} \frac{1}{196608} \sqrt{\frac{2337080251}{10}}.
 \end{aligned}$$

4. Conclusion

In this paper, we introduced some new results of Ostrowski's type inequalities for various norms by using well known inequalities. We also examined modified results. Notably, we developed a new peano kernel i.e., 13-step quadratic kernel. Lastly, we implemented our outcomes to the domain of cumulative distributive functions. In future, one can extend this work by using n-step kernel for different norms and for the function of bounded variation.

6. References

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