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REPORT

On Separability Properties of Nilpotent Groups

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(**Keywords:** *Pereskia bleo,* anti-proliferative, antioxidant, oxidant-induced cell death, apoptosis.)

1.0 INTRODUCTION

In this note we will study two group theoretic properties related to the solution of the Generalised Word Problem, that is, subgroup separability and double coset separability. More precisely we will examine these two properties in extensions of finitely generated nilpotent group. We then apply them to give new proofs that finite extensions of finitely generated nilpotent group are subgroup separable and double coset separable. In general, finite extensions of double coset separable groups are always double coset separable. Our main results are contained in Theorem 3.1 and Theorem 3.3.

We will begin our study with double coset separability in Section 3. The notations used here are standard. In addition the following notations will be used for any group G: $N \lhd_f G$ means N is a normal subgroup of finite index in G.

2.0 PRELIMINARIES.

In this section we recall the definitions of subgroup separability and double coset separability.

Definition 2.1.
A group G is called subgroup separable if for each finitely generated subgroup M and for each $x \in G \backslash M$, there exists $N \lhd_f G$ such that $x \notin MN$.

Clearly a subgroup separable group is residually finite. It is well known that free groups, polycyclic groups (and hence finitely generated nilpotent groups) and surface groups are subgroup separable [1], [6], and [7]). Nevertheless very few other classes of groups are known to be subgroup separable. On the other hand, finite extensions of subgroup separable groups are again subgroup separable.

Definition 2.2.
A group G is called double coset separable if for each pair of subgroups K, M and for each pair of elements $x, y \in G$, such that $x \notin KyM$, there exists $N \lhd_f G$ such that $x \notin KyMN$.

Clearly double coset separability implies subgroup separability. Stebe [8] first showed that finitely generated nilpotent groups are double coset separable. This was extended by Toh [10] to finite extension of nilpotent groups. The property of double coset separability has been used extensively in the study of conjugacy separability and the solution of the Conjugacy Problem in generalized free products, tree products and HNN extensions.[2], [3], [4], [5], [9].

3.0 DOUBLE COSET SEPARABILITY.

In this section we prove that a finite extension of a finitely generated nilpotent group is double coset separable. We begin with the following theorem.

Theorem 3.1.

Let G be an extension of a finitely generated nilpotent group H. Suppose $x, y \in G$ and K_1, K_2 are subgroups of H. If $x \notin K_1yK_2$, then there exists $N \triangleleft_f H$ such that $N \triangleleft G$ and $x \notin K_1yK_2N$.

Proof. We use induction on the torsion free rank of H. If H is finite, then $N = \{1\}$ is the required subgroup. Suppose H is infinite. Let $Z = Z(H)$ be the center of H. Then Z^m is a characteristic subgroup of H and $Z^m \triangleleft G$. By taking a suitably large m, Z^m is a finitely generated free abelian group.

Let $\bar{G}_m = G/Z^m$ and $\bar{H}_m = H/Z^m$. Then the torsion free rank of $\bar{H}_m$ is less than the torsion free rank of H. If $\bar{x} \notin \bar{K}_1y\bar{K}_2$ for some m, then by the induction hypothesis there exists $\bar{N} \triangleleft_f \bar{H}_m$ such that $\bar{N} \triangleleft \bar{G}_m$ and $\bar{x} \notin \bar{K}_1y\bar{K}_2\bar{N}$. Let N be the preimage of $\bar{N}$. Then N is the required subgroup.

Suppose $\bar{x} \in \bar{K}_1y\bar{K}_2$ in $\bar{G}_m = G/Z^m$ for each m. We shall show that this case does not occur. Now from $\bar{G}_1 = G/Z$, we have $x = uyva$ for some $a \in Z$, $u \in K_1$, $v \in K_2$. Note that $K_1yK_2 = K_1uyvK_2$. Therefore we may assume that $x = ya$. Now from each $\bar{G}_m = G/Z^m$, $m \geq 2$, we have

(1) $ya = u_m y v_m c_m, \qquad c_m \in Z^m,\ u_m \in K_1, v_m \in K_2.$

Let $S = gp\{y^{-1}u_m y v_m\}$ where $u_m \in K_1$, $v_m \in K_2$, $m \geq 2$, are as in (1), i.e., S is the subgroup generated by the elements of the form $y^{-1}u_m y v_m$ where $u_m \in K_1$, $v_m \in K_2$, $m \geq 2$ are as in (1). It is not hard to see that $S \subseteq Z$, for from (1), $y^{-1}u_m y v_m \in Z$. Now we will show that $(y^{-1}u_m y v_m)^{-1} = y^{-1}u_m^{-1} y v_m^{-1}$. Let $y^{-1}u_m y v_m = z$ for some $z \in Z$. Then $y^{-1}u_m^{-1}y = z^{-1}v_m$, i.e., $y^{-1}u_m^{-1} y v_m^{-1} = z^{-1} = (y^{-1}u_m y v_m)^{-1}$. This implies that $(y^{-1}u_m y v_m)^{-1} \in S$. Therefore $y^{-1}u_m^{\varepsilon_m} y v_m^{\varepsilon_m} \in Z$ for $\varepsilon_m = \pm 1$ and $\left(y^{-1}u_{m_1}^{\varepsilon_{m_1}} y v_{m_1}^{\varepsilon_{m_1}}\right)\left(y^{-1}u_{m_2}^{\varepsilon_{m_2}} y v_{m_2}^{\varepsilon_{m_2}}\right) = y^{-1}u_{m_1}^{\varepsilon_{m_1}}u_{m_2}^{\varepsilon_{m_2}} y v_{m_1}^{\varepsilon_{m_1}} v_{m_2}^{\varepsilon_{m_2}}$, $\varepsilon_{m_i} = \pm 1$. Hence the elements in S are of the form

(2) $y^{-1}u_{m_1}^{\varepsilon_{m_1}} \ldots u_{m_n}^{\varepsilon_{m_n}} y v_{m_1}^{\varepsilon_{m_1}} \ldots v_{m_n}^{\varepsilon_{m_n}}, \varepsilon_{m_i} = \pm 1.$

Now if $a \in S$, then from (2), $a = y^{-1}u_{m_1}^{\varepsilon_{m_1}} \ldots u_{m_n}^{\varepsilon_{m_n}} y v_{m_1}^{\varepsilon_{m_1}} \ldots v_{m_n}^{\varepsilon_{m_n}}$. But this implies that $x = ya \in K_1yK_2$ since $u_{m_i} \in K_1$ and $v_{m_i} \in K_2$, a contradiction. Therefore $a \notin S$. Since Z is a finitely generated abelian group there exists $M \triangleleft_f Z$ such that $a \notin SM$. Note that for suitably large m, $Z^m \subseteq M$. But from (1), we have $a = y^{-1}u_m y v_m c_m$ and hence $a \in SM$ a contradiction. The proof is now completed.

Lemma 3.2.

[11, Lemma 3.1] *Let H be a finitely generated group and $S \triangleleft_f H$. Then there exists $f_H(S) \subset S$ such that $f_H(S)$ is a characteristic subgroup of finite index in H.*

Theorem 3.3.

Let G be an extension of a finitely generated nilpotent-by-finite group H. Suppose $x, y \in G$ and K_1, K_2 are subgroups of H. If $x \notin K_1yK_2$, then there exists $N \triangleleft_f H$ such that $N \triangleleft G$ and $x \notin K_1yK_2N$.

Proof. Let $M \triangleleft_f H$ such that M is finitely generated nilpotent. Since H is finitely generated, by Lemma 3.2, there exists $f_H(M) \subseteq M$ such that $f_H(M)$ is a characteristic subgroup of finite index in H. Therefore $f_H(M) \triangleleft G$ and $f_H(M)$ is finitely generated nilpotent. So we may assume $M \triangleleft G$ and M is finitely generated nilpotent.

Let $\{u_1, u_2, \ldots, u_{k_1}\}$ be a complete set of left coset representatives of $K_1 \cap M$ in K_1 and $\{v_1, v_2, \ldots, v_{k_2}\}$ be a complete set of right coset representatives of $K_2 \cap M$ in K_2. Note that $x \notin u_i(K_1 \cap M)y(K_2 \cap M v_j)$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. By Theorem 3.1, there exists $N_{ij} \triangleleft_f M$ such that $N_{ij} \triangleleft G$ and $u_i^{-1}xv_j^{-1} \notin (K_1 \cap M)y(K_2 \cap M)N_{ij}$. Let $N = \bigcap_{i,j} N_{ij}$. Then $N \triangleleft_f H$ and $N \triangleleft G$. If $x \in K_1yK_2N$,, then $x \equiv uyv \bmod N$ for some $u \in K_1$ and $v \in K_2$. But $u = u_iu'$ and $v = v'v_j$ for some $u' \in K_1 \cap M$ and $v' \in K_2 \cap M$. This implies that $x \equiv u_iu'yv'v_j \bmod N$ or equivalent to

$u_i^{-1}xv_j^{-1} \in (K_1 \cap M)y(K_2 \cap M)N_{ij}$, a contradiction. The proof is completed.

Corollary 3.4.
Let H be a finite extension of a finitely generated nilpotent group. Then H is double coset separable.

Proof. Put $G = H$ in Theorem 3.3.

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